

# Magnetic Solutions to 2+1 Gravity

Eric W. Hirschmann  
*Department of Physics*  
*University of California*  
*Santa Barbara, CA 93106-9530*  
*ehirsch@dolphin.physics.ucsb.edu*

Dean L. Welch  
*Department of Physics*  
*University of California*  
*Santa Barbara, CA 93106-9530*  
*dean@cosmic.physics.ucsb.edu*

## Abstract

We report on a new solution to the Einstein-Maxwell equations in 2+1 dimensions with a negative cosmological constant. The solution is static, rotationally symmetric and has a non-zero magnetic field. The solution can be interpreted as a monopole with an everywhere finite energy density.

## 1. Introduction

Studying physics in spacetimes with dimensions less than four has often proved useful. While the study of such systems is of intrinsic interest, one usually has the hope that the properties of these lower dimensional systems will mimic the properties of some corresponding four-dimensional system. Witten's discovery [1] of a black hole solution

in two-dimensional string theory has sparked renewed interest in lower dimensional gravity. This solution has been used to study problems which have been intractable in four dimensions such as black hole information loss.

Another lower-dimensional black hole solution that has generated a great deal of interest is the three-dimensional black hole discovered by Bañados, Teitelboim and Zanelli (BTZ) [2] [3]. This spacetime is a solution to Einstein gravity with a negative cosmological constant, it is also known that this solution can be formulated as a string theory solution [4]. Like the two dimensional black hole, this solution has been studied with the hope of shedding light on problems in four-dimensional gravity. This hope is supported by the fact that there are striking similarities between some of these recently discovered three-dimensional solutions and their four dimensional counterparts. However, despite these similarities one should bear in mind that, there are some important differences, not the least of which is that the universe we live in is four dimensional. Nevertheless, in the more simplified realm of three dimensions we might reasonably hope to obtain some insight into the nature of gravity and quantum gravity in particular [5].

The discovery of the BTZ black hole has spawned efforts to find other solutions to the three-dimensional Einstein equations as well as solutions to various generalizations of them coupled to a variety of matter fields. One such solution is the static electrically charged black hole originally discussed by BTZ [2]. This solution is specified by three parameters, a mass parameter  $M$ , a charge  $Q$ , and a “radial parameter”  $r_0$ . To see that one needs this radial parameter it is sufficient to observe that while the energy density in the electromagnetic field approaches zero asymptotically, the rate at which it does is sufficiently slow so that the total energy in the electromagnetic field outside of any finite radius diverges. This is most easily seen by observing the behavior of the quasilocal mass as a function of  $r$ ,  $M_{ql} = M + Q^2 \ln(r/r_0)$ . The parameter  $r_0$  serves to determine how much of the energy in the electromagnetic field is included in the mass parameter  $M$ .<sup>1</sup> Depending on the values of the parameters the charged BTZ solutions may have two, one, or no horizons. It is natural to identify a charged solution with a single horizon as an extremal black hole. This identification is supported by the observation that such a solution has zero Hawking temperature, as does the extremally charged Reissner-Nordstrom black hole.

Although the static, electrically charged solution is similar in many ways to the Reissner-Nordstrom black hole in four dimensions, there are some important differences.

---

<sup>1</sup> In other words, for a given solution, changes in  $M$  can be compensated for by changes in  $r_0$ , see [2].

The most obvious difference is that the three-dimensional black hole is asymptotically anti-de Sitter space while the Reissner-Nordstrom solution is asymptotically flat. Another difference that we have just seen is that the static solution has a quasilocal mass that diverges at infinity whereas the quasilocal mass of the four-dimensional charged black hole approaches a constant asymptotically [6]. An additional difference that is the consideration of this work is that the Reissner-Nordstrom black hole can have electric or magnetic charge, as well as both. Due to the invariance of the Maxwell equations under a duality transformation, the form of the metric for a Reissner-Nordstrom black hole is the same for an electrically charged solution and a magnetically charged solution. The reason is that in four dimensions both the Maxwell tensor and its dual are two-forms. However, no such transformation exists for the Maxwell equations in three dimensions because the Maxwell tensor is a two-form and its dual is a one-form. One is naturally led to ask whether the solutions to the Einstein-Maxwell equations in three dimensions are different if one assumes that they possess a magnetic as opposed to electric charge. We examine this question in this paper and report that the solutions are quite different. Whereas the electric solution may be a black hole provided the charge is not too large, the magnetic solution that we present is not black hole for any value of the magnetic charge. This magnetic solution is both static and rotationally symmetric. In addition, it has finite energy density. We interpret it as a magnetic monopole.<sup>2</sup>

Parenthetically, we remark that there has been some recent discussion of stationary generalizations of the electrically charged solution [7] [8]. For a rotating and charged solution, one would expect there to be both an electric and a magnetic field. Indeed, these new rotating solutions would appear to possess both. However, it was incorrectly reported in [7] that their extremal solution has a finite angular momentum. Chan [9] showed that the angular momentum of the solution in [7] actually diverges logarithmically at infinity. Given that the mass of a static electrically charged solution also diverges logarithmically at infinity, we believe that the divergence in the angular momentum is not physically unreasonable.

---

<sup>2</sup> We are using the term monopole a bit loosely here. The solution is certainly magnetic and particle-like, but the fact that we are in two spatial dimensions suggests that the solution is perhaps a bit more reminiscent of a Neilson-Olesen vortex solution.

## 2. The Einstein Equations and Their Solution

We begin with the action for Einstein gravity coupled to a  $U(1)$  gauge field with a negative cosmological constant.

$$S = \frac{1}{4} \int d^3x \sqrt{-g} (R - 2\Lambda - F^2) \quad (2.1)$$

where  $\Lambda = -1/l^2$  is the negative cosmological constant,  $F$  is a two-form and we have set Newton's constant to be  $1/4\pi$ . The equations of motion derived from the action are the Einstein equations

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = 2T_{ab} \quad (2.2)$$

and the Maxwell equations

$$\nabla_a F^{ab} = 0 \quad (2.3)$$

with the stress tensor of the electromagnetic field given by

$$T_{ab} = F_{ac}F_{bd}g^{cd} - \frac{1}{4}g_{ab}F^2. \quad (2.4)$$

We assume the spacetime is both static and rotationally symmetric, implying the existence of a timelike Killing vector and a spacelike Killing vector. In the coordinate basis we use, these vectors will be  $\partial/\partial t$  and  $\partial/\partial \phi$  respectively. These symmetries allow us to write our metric in the following form [7]

$$ds^2 = -N(r)^2 dt^2 + L(r)^{-2} dr^2 + K(r)^2 d\phi^2. \quad (2.5)$$

Using this metric and the substitutions  $E_r = \frac{L}{N}F_{tr}$  and  $B = \frac{L}{K}F_{r\phi}$ , ( $E_r$  and  $B$  are the components of the Maxwell tensor measured in an orthonormal basis) the Einstein-Maxwell equations, (2.2) through (2.4) become

$$\begin{aligned} R_{tt} &= N^2 L^2 \left( \frac{N'K'}{NK} + \frac{L'K'}{LK} + \frac{N''}{N} \right) \\ &= N^2 \left( \frac{2}{l^2} + 2B^2 \right) \end{aligned} \quad (2.6a)$$

$$\begin{aligned} R_{\phi\phi} &= -K^2 L^2 \left( \frac{N'K'}{NK} + \frac{L'K'}{LK} + \frac{K''}{K} \right) \\ &= K^2 \left( -\frac{2}{l^2} + 2E_r^2 \right) \end{aligned} \quad (2.6b)$$

$$\begin{aligned} R_{t\phi} &= 0 \\ &= -2E_r B \end{aligned} \quad (2.6c)$$

$$\begin{aligned} G_{rr} &= \frac{N'K'}{NK} \\ &= \frac{1}{L^2} \left( \frac{1}{l^2} + B^2 - E_r^2 \right) \end{aligned} \quad (2.6d)$$

where the prime indicates differentiation with respect to  $r$ . In addition, we can write the Maxwell equation as

$$\partial_a(\sqrt{-g}g^{ab}g^{cd}F_{bc}) = 0 \quad (2.7)$$

which, upon integration yields

$$E_r = \frac{C_1}{K}, \quad B = \frac{C_2}{N}. \quad (2.8)$$

We have made no assumptions other than the fact that the spacetime is static and rotationally symmetric. The  $R_{t\phi}$  equation implies that one or both of the electric and magnetic fields must be zero.<sup>3</sup> The electric case has previously been discussed. However, we are interested in magnetically charged solutions, so we make the assumption that  $C_2 \neq 0$  which immediately implies that  $C_1 = 0$ .

Using our form for  $B$ , we can solve our equations as follows. We make the substitution

$$L(r) = KNf(r) \quad (2.9)$$

where  $f(r)$  is a function which can be freely specified. We can combine (2.6a) and (2.6d) to get an equation in  $N$  and  $f$

$$\frac{f}{2}(N^2)' = a_0 \quad (2.10)$$

where  $a_0$  is an integration constant. Likewise, equation (2.6d) will now yield

$$\frac{f}{2}(K^2)' = \frac{1}{l^2} + \frac{2C_2^2}{N^2}. \quad (2.11)$$

For the simple choice  $f(r) = 1/r$ , the metric coefficients become

$$N^2(r) = a_0 r^2 + a_1 \quad K^2(r) = \frac{1}{l^2 a_0} r^2 + \frac{C_2^2}{a_0^2} \ln |a_0 r^2 + a_1| + a_2 \quad (2.12)$$

This solution is asymptotic to anti-de Sitter space with curvature  $-1/l^2$ . The integration constant  $a_2$  can be absorbed into the other integration constants together with a rescaling of  $r^2$ , so we will choose it to be zero. We choose a normalization of  $t$  so that as  $r$  becomes large,  $g_{tt}$  approaches  $-r^2/l^2$ . This is equivalent to choosing  $a_0 = 1/l^2$ . When the magnetic field is zero,  $C_2 = 0$ , this solution is a three-dimensional black hole with mass equal to  $-a_1$ . Therefore, we set  $a_1 = -M$ . Asymptotically, the Maxwell field looks like that of a

---

<sup>3</sup> This will no longer be necessarily true for the rotating case.

magnetic point charge, so we set  $C_2 = Q_m/l^2$  ( $Q_m$  representing the magnetic charge). The metric (2.5) is now in the form

$$ds^2 = -(r^2/l^2 - M)dt^2 + r^2(r^2/l^2 - M)^{-1}(r^2 + Q_m^2 \ln|r^2/l^2 - M|)^{-1}dr^2 + (r^2 + Q_m^2 \ln|r^2/l^2 - M|)d\phi^2. \quad (2.13)$$

For future convenience, we make the definition,  $r_+^2 = Ml^2$ . For  $Q_m = 0$ , the metric (2.13) is identical to the nonrotating BTZ solution, as we would expect. However, the presence of a nonzero magnetic charge drastically changes the spacetime.

The nonrotating three-dimensional black hole obtained by setting the magnetic charge to zero has an event horizon at  $r = r_+$ . However, there is no event horizon for the case of nonzero magnetic charge. In particular, we do not have a magnetically charged three-dimensional black hole. This can be seen as follows. The  $g_{\phi\phi} = K^2$  term becomes zero for some value of  $r$  which we call  $\bar{r}$ . By definition  $\bar{r}$  satisfies

$$\bar{r}^2 + Q_m^2 \ln\left|\frac{\bar{r}^2 - r_+^2}{l^2}\right| = 0. \quad (2.14)$$

Clearly  $\bar{r}$  is constrained to be between  $r_+$  and  $\sqrt{r_+^2 + l^2}$ . Not only does  $g_{\phi\phi}$  change sign as  $r$  becomes less than  $\bar{r}$ , but  $g_{rr}$  changes sign as well. One can see that naively using these coordinates for  $r < \bar{r}$  leads to an apparent signature change. This shows that we must choose a different continuation for  $r \leq \bar{r}$  [10]. We now introduce a new set of coordinates that will show the spacetime is complete for  $r \geq \bar{r}$ . A “good” set of coordinates which allows us to cover our spacetime is found by letting

$$x^2 = r^2 - \bar{r}^2$$

The metric with this new coordinate then becomes

$$ds^2 = -\frac{1}{l^2}(x^2 + \bar{r}^2 - r_+^2)dt^2 + (x^2 + Q_m^2 \ln[1 + \frac{x^2}{\bar{r}^2 - r_+^2}])d\phi^2 + l^2 x^2 (x^2 + \bar{r}^2 - r_+^2)^{-1} (x^2 + Q_m^2 \ln[1 + \frac{x^2}{\bar{r}^2 - r_+^2}])^{-1} dx^2 \quad (2.15)$$

where our coordinate  $x$  ranges between zero and infinity. This coordinate system now covers the complete spacetime. Timelike geodesics can reach the origin,  $x = 0$ , in a finite proper time and null geodesics can reach the origin in a finite affine parameter. The components of the Ricci tensor measured in an orthonormal basis that is parallel

propagated along a timelike geodesic are well behaved everywhere. In three dimensions the curvature is completely determined by the Ricci tensor, so the fact that the Ricci tensor is well behaved shows that this spacetime has no curvature singularities. Similarly, the components of the electromagnetic field strength are well behaved in this basis.

However, one can see that at  $x = 0$ , we will have a conical singularity unless we identify the coordinate  $\phi$  with a certain period. The period is found to be

$$T_\phi = 2\pi \frac{e^{\beta/2}}{1 + Q_m^2 e^\beta / l^2} \quad (2.16)$$

where  $\beta = \bar{r}^2 / Q_m^2$ . The strange thing about this period reveals itself when we examine its behavior for limiting values of  $Q_m$ . As  $Q_m$  approaches infinity the period becomes zero. This is what one might expect because this is the limit in which the magnetic charge is approaching infinity. However, as  $Q_m$  approaches zero the same thing happens, the period of the coordinate  $\phi$  approaches zero again. This is very surprising since the  $Q_m = 0$  solution is a three-dimensional black hole (with no magnetic charge) and this looks nothing like the  $Q_m \neq 0$  solution in the limit as  $Q_m$  approaches zero. While it often occurs that “the limit of a theory is not the theory of the limit,” in the case considered here the difference is quite striking.

### 3. The Spacetime for Negative $M$

In the previous analysis we have been assuming that  $M \geq 0$ . We now briefly consider magnetic solutions with negative  $M$ . As observed in [2], the BTZ solution for  $-1 < M < 0$  reduces to a solution with a naked conical singularity. Such solutions were studied in Refs. [11] [12]. For  $M$  in this range the magnetic solution, (2.13) with  $Q_m \neq 0$ , is continued in the same way as for the case of nonnegative  $M$ . However, now we have  $r^2 + l^2 |M|$  appearing in the logarithm in (2.13), so  $\bar{r}$  is constrained to be between 0 and  $l\sqrt{1 - |M|}$ .

For  $M = -1$  the BTZ solution is anti-de Sitter space. It is interesting to observe that the magnetic solution with  $M = -1$  is already complete with no apparent signature change in the metric. Equivalently, the analysis in the previous section applies, but with  $\bar{r} = 0$ . In particular, note that the period of  $\phi$  needed to avoid a conical singularity is

$$T_\phi(M = -1) = \frac{2\pi}{1 + Q_m^2 / l^2} \quad (3.1)$$

for  $M = -1$ . This period still approaches zero as  $Q_m$  approaches infinity, but as  $Q_m$  approaches zero (3.1) goes to a constant. This behavior of the period of  $\phi$  is what intuition tells one it should be (one should bear in mind that the coordinate  $\phi$  is not identified for anti-de Sitter space).

Finally, consider the magnetic solution with  $M < -1$ . This space is incomplete if one only considers non-negative values of  $r^2$ . To complete this space we must allow  $r^2$  to become negative. This is not as strange a thing to do as it may seem (in particular it does not require us to consider complex coordinates). Note that  $r$  in (2.13) only appears as  $r^2$ , this indicates that  $r^2$  may be a more natural radial coordinate (see also the transformations in [4]). The coordinate transformation  $r^2 = x^2 - l^2|M|$  leads to the desired result. If we allow  $x$  to range over all non-negative values the space will be complete. The remainder of the analysis carries through as with the  $M = -1$  case.

#### 4. Conclusions

In this paper we have presented a new solution to the Einstein-Maxwell equations in 2+1 dimensions in the presence of a negative cosmological constant. It is static, rotationally symmetric and magnetically charged. The nature of the spacetime is very different from that for the electrically charged BTZ solution. In the latter case, for a nonzero region of parameter space the solution is a black hole. In contrast, the magnetic case has no event horizon and is particle-like.

There are several other things one might like to know about this solution. One possibility would be to understand the motion of magnetically charged particles in this spacetime. Another interesting question would be whether this solution could be generalized to one that included rotation.

#### Acknowledgments

We wish to thank Gary Horowitz for helpful discussions and Doug Eardley for reading a preliminary draft. This work was supported by NSF Grant No. PHY-9008502.



## References

- [1] E. Witten, *Phys Rev.* **D44**, 314 (1991).
- [2] M. Banados, C. Teitelboim, and J. Zanelli, *Phys. Rev. Lett.* **69**, 1849 (1992).
- [3] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, *Phys. Rev.* **D48**, 1506 (1993).
- [4] G. Horowitz and D. Welch, *Phys. Rev. Lett.* **71**, 316 (1993).
- [5] For a nice review see S. Carlip, hep-th 9506079.
- [6] See for example, Hugget and Todd, *Introduction to Twistor Theory*, second edition, Cambridge University Press, 1994.
- [7] M. Kamata and T. Koikawa, *Phys. Lett.* **B353**, 196 (1995).
- [8] G. Clement, gr-qc 9510025.
- [9] K. Chan, gr-qc 9509032.
- [10] See for example, J. Horne and G. Horowitz, *Nucl. Phys.* **B368**, 444 (1992).
- [11] S. Deser, R. Jackiw, and G 't Hooft, *Ann. Phys.* **152**, 220,1984.
- [12] J.D. Brown, Ph.D. thesis, University of Texas at Austin 1985; published as a book *Lower Dimensional Gravity* (World Scientific, Singapore (1988)).